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On Poisson geometry and supersymmetric sigma models

C. KLIMČÍK

Institut de mathématiques de Luminy,
163, Avenue de Luminy,
13288 Marseille, France
e-mail: ctirad.klimcik@univ-amu.fr

Abstract

By using the Poisson geometry, we develop a manifestly invariant and calculation-friendly formalism for handling $UOSp(2|1)$ -supersymmetric field theories. In particular, the super-Lagrangians are written solely in terms of superfields, Poisson brackets and the moment map generating the $UOSp(2|1)$ action. As an application of this formalism, we construct the Kalb-Ramond term for supersymmetric sigma models on the supersphere.

1 Introduction

Consider a smooth map $\phi : \Sigma \rightarrow T$ where Σ and T are Riemannian manifolds. Such a map is called harmonic if it is a solution of field equations of the so called nonlinear sigma model associated to Σ and T . The Lagrangian of this model is given by the squared double norm (with respect to the metrics g on Σ and G on T) of the derivation of ϕ and the action is obtained by integration of the Lagrangian with respect to the measure $d\mu_g$ on Σ induced by the metric g :

$$\mathcal{S}_G = \int d\mu_g ||d\phi||_{g,G}^2. \quad (1)$$

Obviously, the nonlinear sigma model is symmetric with respect to the group of isometries of the source manifold Σ . For example, if Σ is the two sphere S^2 the model has the rotational $SO(3)$ symmetry.

If Σ is two-dimensional, there exists a generalisation of the nonlinear sigma model considered mainly in string theory which is induced by a presence of an additional geometrical structure on the target T . This structure is called the Kalb-Ramond field [10] and it is nothing but a two-form field B on T . The pull-back ϕ^*B integrated over Σ is then added to the original sigma model action in order to take into account the presence of B :

$$\mathcal{S}_{GB} = \int d\mu_g ||d\phi||_{g,G}^2 + \int \phi^*B. \quad (2)$$

Note that the Kalb-Ramond term $\int \phi^*B$ is not only invariant with respect to the isometries of Σ but it is invariant even with respect to all diffeomorphisms of Σ .

If E is a Riemannian supermanifold with the bosonic body being the flat Euclidean 2-plane and odd coordinates being $\xi, \bar{\xi}$ then there exists a supersymmetric generalisation of the nonlinear sigma model which includes the Kalb-Ramond term [6, 7]. Its action in the form of the Berezin integral reads

$$S_E = \int d\bar{z}dzd\bar{\xi}d\xi(G_{IJ}(Y^K) + iB_{IJ}(Y^K))DY^I\bar{D}Y^J, \quad (3)$$

where Y^I are the superfields (i.e. even functions on the Euclidean superplane) corresponding to the coordinates on the target space and G_{IJ} and B_{IJ} are respectively the components of the target space metric and the Kalb-Ramond field in those coordinates. Moreover, the supersymmetric covariant derivatives are defined as

$$\bar{D} := \partial_{\bar{\xi}} + \bar{\xi}\partial_{\bar{z}}, \quad D := \partial_{\xi} + \xi\partial_z. \quad (4)$$

Now the Kalb-Ramond part of the supersymmetric action (3) is a less geometric object as in the bosonic case since it can no longer be written in terms of a pull-back Y^*B by a sigma model superfield Y . Indeed, though B is the two-form, the volume form on the source supermanifold E is not a two-form due to the presence of odd differentials. As the result, the Kalb-Ramond term is not supersymmetric with respect to all superdiffeomorphisms of E but just with respect to the superisometries of E . In fact, the supersymmetric Kalb-Ramond term is *determined* by the

criterion of superinvariance with respect to the Euclidean superisometries and by the criterion that, when the superfields Y^I do not depend on the odd coordinates $\bar{\xi}, \xi$, the supersymmetric action (3) must reduce to the bosonic action (1).

String theoretical motivations caused recently a growing interest in formulations of rigidly supersymmetric field theories on curved space-times [1, 2, 3, 4, 8, 9, 11, 13, 14, 16, 17]. It is therefore a natural question what happens to the Kalb-Ramond term in the curved context. The answer is simple: this term is to be determined by the same two conditions as in the flat case i.e. it must be invariant with respect to the superisometries of the curved worldsheet and it must reduce to the geometric term $\int \phi^* B$ in the bosonic limit. In this paper, we shall study the Kalb-Ramond term for sigma models on the so called supersphere¹ $S^{2|2}$ which is the simplest supersymmetrization of the standard sphere S^2 with the supergroup of superisometries being the unitary orthosymplectic supergroup $UOSp(2|1)$.

In fact, to construct the action of the $UOSp(2|1)$ supersymmetric nonlinear sigma model on the curved super-worldsheet $S^{2|2}$ is far from being just a straightforward generalisation of the flat super-Euclidean situation. Indeed, although it is quite straightforward to construct various differential supersymmetric invariants of the rigid supersymmetry supergroup, it is a fairly less trivial task to work out which invariants give rise to viable field theories. The issue is that seemingly "nice" invariant action principle written in the superfield formalism may be in fact pathological when worked out in components. Typically, there may occur a violation of spin-statistics (a presence of quadratic bosonic derivatives in the fermionic kinetic term) and also other unwanted phenomena (like fourth order bosonic derivatives in the case the theory contains a gauge symmetry). Clues to select non-pathological candidates vary from case to case and no universal algorithms are available.

The problem of finding the sigma model action on the supersphere was posed already in [5] but it is fully solved only in the present article since the old work [5] and the subsequent work [18] constructed the supersymmetric sigma model on $S^{2|2}$ without the Kalb-Ramond term. The main reason why the $uosp(2|1)$ version of the Kalb-Ramond term was not constructed in [5] was exceeding technical difficulty in working out which of all possible $uosp(2|1)$ supersymmetric invariants leads to a non-pathological theory with a correct bosonic limit. The decisive technical progress reported in this paper is due to a new compact formalism based on the Poisson brackets and moment maps. This formalism significantly streamlines and facilitates the technical work needed to check the viability of tentative invariant Lagrangians. In particular, the verification of supersymmetry is a one-line check, e.g. (60) and the rapid calculation of the bosonic limit is based on the moment map identities (65).

Without anticipating all details, we find that the correct $UOSp(2|1)$ invariant supersymmetric sigma model action on the supersphere including the Kalb-Ramond term turns out to be

$$S_{sGB} = -\text{STr} \int d\mu_{S^{2|2}} (G_{IJ} + 2i\mathcal{M}B_{IJ}) \{\mathcal{M}^2, Y^I\} \{\mathcal{M}^2, Y^J\}. \quad (5)$$

Here (the moment map) \mathcal{M} is a fixed $uosp(2|1)$ -matrix valued function on $S^{2|2}$, $\{.,.\}$ stands for the Poisson bracket and $d\mu_{S^{2|2}}$ is an $uosp(2|1)$ -invariant measure on $S^{2|2}$. We note the ap-

¹Note that the supersphere itself can be understood as the coset supermanifold $UOSp(2|1)/U(1)$.

pearance in the action (5) of the both moment map \mathcal{M} , which encompasses all supersymmetric generators and its square \mathcal{M}^2 , which turns out to encompass all supersymmetric covariant derivatives. It is precisely this circumstance that illustrates that from the structural point of view the supersymmetric action (5) is not quite a direct generalisation of the purely bosonic $SO(3)$ invariant sigma model the action of which reads

$$S_{GB} = \frac{1}{2} \text{Tr} \int d\mu_{S^2} (G_{IJ} + iM B_{IJ}) \{M, Y^I\} \{M, Y^J\}. \quad (6)$$

Here the bosonic moment map M generates the $SO(3)$ symmetry and its square does not appear in the story (in fact, unlike the moment map \mathcal{M} , M squares to the unit matrix!). In spite of differences, the supersymmetric action (5) will be shown to reduce to the bosonic action (6) when the fermionic parts of the superfields Y^I are set to zero.

In a short Section 2, we review the concept of a Hermitian supermatrix and then in Section 3 we describe features of the Poisson geometries of the sphere S^2 and of the supersphere $S^{2|2}$. Finally, in Section 4, we first construct the ordinary $SO(3)$ invariant bosonic sigma model (6) on the ordinary sphere S^2 , then the $UOSp(2|1)$ -invariant super sigma model (5) on the supersphere $S^{2|2}$ and we establish that in the absence of the fermions the superaction (5) does reduce to the bosonic action (6).

2 Supermatrices

Consider a complex Grassmann algebra G equipped with a \mathbb{C} -antilinear map call graded conjugation [15], which associates to every $a \in G$ an element $\bar{a} \in G$ in such a way that

$$\overline{ab} = \bar{a}\bar{b}, \quad \bar{\bar{a}} = (-1)^{p(a)}a, \quad a, b \in G. \quad (7)$$

Here $p(a)$ means the Grassmann parity of a . By a supermatrix² we mean a square matrix M with a distinguished parities of indices for which the Grassmann parity of an element $M_{ij} \in G$ is the same as the sum $p(i) + p(j)$ of the index parities. Moreover, the elements M_{ij} of a "Hermitian supermatrix" satisfy the relation

$$M_{ij} = \bar{M}_{ji}, \quad i \geq j. \quad (8)$$

Note that in the purely bosonic case (8) remains true for all indices i, j , however in the supercase the restriction to the inequality $i \geq j$ is essential.

The supertrace $\text{STr}(M)$ of a supermatrix M is defined as

$$\text{STr}(M) := \sum_i (-1)^{p(i)} M_{ii}. \quad (9)$$

There is a natural supermeasure $d\mu_H$ on the superhermitian matrices given by the formula

$$d\mu_H := \Pi_i dM_{ii} \Pi_{i < j} dM_{ij} d\bar{M}_{ij}, \quad (10)$$

²In this paper we shall not consider "odd" supermatrices for which the Grassmann parity of an element M_{ij} is opposite to the sum $p(i) + p(j)$ of the index parities

where $dM_{ij}d\bar{M}_{ij}$ is the Berezin measure if $p(i) + p(j)$ is odd and the standard Lebesgue measure on $\mathbb{C} = \mathbb{R}^2$ if $p(i) + p(j)$ is even.

The purely "bosonic" case corresponds to the situation where all index parities are even. All formulae presented in this section remains then true just the terminology flips e.g. from the supertrace STr to the ordinary trace Tr etc.

3 Sphere and supersphere

3.1 Sphere S^2

We describe the ordinary two-dimensional sphere in the way best suited for the later supersymmetric generalisation. Thus we define the sphere S^2 as the set of ordinary (purely bosonic) Hermitian 2×2 matrices M such that

$$\text{Tr}(M) = 0, \quad \text{Tr}(M^2) = 2. \quad (11)$$

Indeed, in terms of the matrix components M_{ij} the first condition gives

$$M_{11} = -M_{22} \quad (12)$$

and the second one

$$M_{11}^2 + \bar{M}_{12}M_{12} = 1. \quad (13)$$

We shall view the Hermitian matrices M verifying the conditions (11) as points on the sphere but the matrix elements M_{ij} as particular *functions* on the sphere. The algebra $C^{pol}(S^2)$ generated by M_{ij} is then a (dense) subspace of the space $C^\infty(S^2)$ of smooth complex functions on S^2 .

The sphere (e.g. viewed as the surface of the unit ball in the three-dimensional Euclidean space) can be naturally rotated by the group $SO(3)$. The infinitesimal generators $V \in so(3)$ of this action turn out to act on the point M of the sphere as $i[V, M]$, where V is viewed as a traceless Hermitian matrix. (The Lie commutator on $so(3)$ is then i -multiple of the matrix commutator.)

A natural Poisson bracket on $C^{pol}(S^2)$ is defined by the following formula

$$\{\text{Tr}(UM), \text{Tr}(VM)\} := -i\text{Tr}([U, V]M), \quad (14)$$

where U, V are any constant traceless Hermitian matrices and $[U, V]$ is the standard matrix commutator. If we choose the basis in $so(3)$ in terms of the standard Pauli matrices σ_m then Eq. (14) can be written equivalently as

$$\{\text{Tr}(\sigma_j M), \text{Tr}(\sigma_k M)\} = -i\text{Tr}([\sigma_j, \sigma_k]M) = 2\varepsilon_{jkm}\text{Tr}(\sigma_m M), \quad (15)$$

where ε_{jkm} is the standard alternating symbol. If we moreover define

$$x_m \equiv \frac{1}{2}\text{Tr}(\sigma_m M), \quad (16)$$

then Eq. (15) becomes

$$\{x_j, x_k\} = \varepsilon_{jkm} x_m. \quad (17)$$

Note also that from the definition (16) it follows

$$M = x_m \sigma_m = \begin{pmatrix} x_3 & x_1 - ix_2 \\ x_1 + ix_2 & -x_3 \end{pmatrix} \quad (18)$$

and the condition (13) becomes

$$x_1^2 + x_2^2 + x_3^2 = 1. \quad (19)$$

We thus observe that x_m are the fixed "Cartesian" functions on the sphere defined by the standard embedding of S^2 into the Euclidean space \mathbb{R}^3 .

The defining brackets (14) can be rewritten equivalently as

$$\{\text{Tr}(UM), M\} = i[U, M]. \quad (20)$$

It is easy to verify that (14) indeed determines a Poisson bracket, in particular, the Poisson Jacobi identity is the consequence of the matrix Jacobi identity. Moreover, by taking the trace of (20) and of $\{\text{Tr}(UM), M^2\} = i[U, M^2]$, we derive

$$\{M, \text{Tr}(M)\} = \{M, \text{Tr}(M^2)\} = 0, \quad (21)$$

which is obviously needed for consistence with the definition (11) of the sphere.

Looking at (20), we immediately see that the $so(3)$ -action is Hamiltonian with respect to the Poisson structure $\{., .\}$. The corresponding moment map is clearly M and the Hamiltonian corresponding to the $so(3)$ generator U is $\text{Tr}(UM)$. This can be seen also from (17). Indeed, denote \mathcal{R}_j the vector field generating the infinitesimal rotations of S^2 around the j axis. It is well-known that

$$\mathcal{R}_j x_k = \varepsilon_{jkm} x_m, \quad (22)$$

hence, from (17)

$$\mathcal{R}_j x_k = \{x_j, x_k\}. \quad (23)$$

Since, moreover, x_k are the generating functions of all spherical harmonics, we have for every smooth function f on the sphere

$$\mathcal{R}_j f = \{x_j, f\}. \quad (24)$$

It follows that the Poisson structure (14) is $so(3)$ invariant:

$$\{\text{Tr}(UM), \{f, g\}\} = \{\{\text{Tr}(UM), f\}g\} + \{f, \{\text{Tr}(UM), g\}\}, \quad \forall f, g \in C^{pol}(S^2). \quad (25)$$

A natural round measure on the sphere S^2 can be defined with the help of the measure $d\mu_H$ on Hermitian matrices weighted by delta functions of the constraints which define the sphere:

$$d\mu_{S^2} := d\mu_H \delta(\text{Tr} M) \delta\left(\frac{1}{2} \text{Tr} M^2 - 1\right) = dx_1 dx_2 dx_3 \delta(x_1^2 + x_2^2 + x_3^2 - 1). \quad (26)$$

We now wish to check, that this measure $d\mu_{S^2}$ is indeed rotational invariant. For that it is sufficient to check the invariance of $d\mu_H$ since the invariance of the arguments of the delta

functions follows from (21). The infinitesimal change of coordinates induced by the rotation V is obviously

$$\delta M = i\varepsilon[V, M] \equiv i\varepsilon \text{Ad}_V M \quad (27)$$

where ε is a small parameter. The induced Jacobian is then

$$\det(1 + i\varepsilon \text{Ad}_V) = 1 + i\varepsilon \text{Tr}(\text{Ad}_V) = 1, \quad (28)$$

which means that the measure is indeed invariant.

The immediate consequence of the invariance of the measure $d\mu_{S^2}$ is the formula

$$\int d\mu_{S^2} \{M, f\} = 0, \quad \forall f \in C^{pol}(S^2), \quad (29)$$

since $\{M, f\}$ is the (matrix valued) variation of the function f under (all possible) infinitesimal rotations.

3.2 Supersphere $S^{2|2}$

A 3×3 Hermitian supermatrix \mathcal{V} with two even indices 1, 2 and one odd index 3 is called orthosymplectic, if it satisfies

$$\mathcal{V}_{33} = 0, \quad \mathcal{V}_{23} = \bar{\mathcal{V}}_{13}. \quad (30)$$

An i-multiple of the standard commutator of two orthosymplectic supermatrices is again orthosymplectic and the corresponding (unitary orthosymplectic) Lie superalgebra is referred to as $uosp(2|1)$.

We now define the supersphere (or rather the algebra $C^{pol}(S^{2|2})$ of polynomial functions on the supersphere) in a more invariant way than in [5], namely, we view it as the algebra generated by matrix elements of a Hermitian orthosymplectic supermatrix \mathcal{M} submitted to further constraints

$$\text{STr}(\mathcal{M}) = 0, \quad \text{STr}(\mathcal{M}^2) = 2. \quad (31)$$

Equivalently, solving the linear constraints give five independent generators which must verify the remaining quadratic constraint

$$\mathcal{M}_{11}^2 + \mathcal{M}_{12}\bar{\mathcal{M}}_{12} + 2\mathcal{M}_{13}\bar{\mathcal{M}}_{13} = 1. \quad (32)$$

Note, that if the odd generators $\mathcal{M}_{13}, \bar{\mathcal{M}}_{13}$ vanish then (32) reduces to the defining relation (13) of the ordinary sphere.

The supersphere can be "superrotated" by the unitary orthosymplectic group $UOSp(2|1)$ the Lie superalgebra of which is $uosp(2|1)$. Infinitesimal action of $\mathcal{V} \in uosp(2|1)$ is just given by $i[\mathcal{V}, \mathcal{M}]$. This action is Hamiltonian (with the Hamiltonian equal to $\text{STr}(\mathcal{V}\mathcal{M})$ and the moment map equal to $\mathcal{M} \in uosp(2|1)$) if we define an $uosp(2|1)$ invariant Poisson structure on $C^{pol}(S^{2|2})$ by the bracket

$$\{\text{STr}(\mathcal{U}\mathcal{M}), \text{STr}(\mathcal{V}\mathcal{M})\} := -i\text{STr}([\mathcal{U}, \mathcal{V}]\mathcal{M}), \quad \mathcal{U}, \mathcal{V} \in uosp(2|1). \quad (33)$$

We may choose a basis of $uosp(2|1)$ as $\Sigma_1, \Sigma_2, \Sigma_3, \Sigma_+ + \Sigma_-, i\Sigma_+ - i\Sigma_-$ in terms of the 3×3 supermatrices Σ_A :

$$\Sigma_1 := \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \Sigma_2 := \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \Sigma_3 := \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad (34)$$

$$\Sigma_+ := \begin{pmatrix} 0 & 0 & \chi \\ 0 & 0 & 0 \\ 0 & -\chi & 0 \end{pmatrix}, \quad \Sigma_- := \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & \bar{\chi} \\ \bar{\chi} & 0 & 0 \end{pmatrix}. \quad (35)$$

Here $\chi, \bar{\chi}$ are auxiliary Grassmann variables which are needed just to ensure the compatibility with our definition of the supermatrix in Section 2 and which will disappear from the explicit Poisson brackets (37) and (38).

If we moreover define

$$x_m \equiv \frac{1}{2} \text{STr}(\Sigma_m \mathcal{M}), \quad m = 1, 2, 3, \quad \chi \theta_+ \equiv \frac{1}{2} \text{STr}(\Sigma_+ \mathcal{M}), \quad \bar{\chi} \theta_- \equiv \frac{1}{2} \text{STr}(\Sigma_- \mathcal{M}) \quad (36)$$

then Eq. (33) gives the supersymmetric analogue of (17):

$$\{x_j, x_k\} = \varepsilon_{jkm} x_m, \quad \{x_3, \theta_{\pm}\} = \mp \frac{1}{2} i \theta_{\pm}, \quad \{x_1 \pm i x_2, \theta_{\pm}\} = 0, \quad \{x_1 \pm i x_2, \theta_{\mp}\} = -i \theta_{\pm}, \quad (37)$$

$$\{\theta_{\pm}, \theta_{\pm}\} = \mp \frac{1}{2} i (x_1 \pm i x_2), \quad \{\theta_{\pm}, \theta_{\mp}\} = \frac{1}{2} i x_3. \quad (38)$$

Note also that from the definition (36) it follows the supersymmetric analogue of (18):

$$\mathcal{M} = \begin{pmatrix} x_3 & x_1 - i x_2 & -\theta_- \\ x_1 + i x_2 & -x_3 & \theta_+ \\ \theta_+ & \theta_- & 0 \end{pmatrix} \quad (39)$$

Now (33) clearly implies

$$\{\text{STr}(\mathcal{V} \mathcal{M}), \mathcal{M}\} = i[\mathcal{V}, \mathcal{M}]. \quad (40)$$

It can be also easily checked that it holds

$$\{\mathcal{M}, \text{STr}(\mathcal{M})\} = \{\mathcal{M}, \text{STr}(\mathcal{M}^2)\} = 0 \quad (41)$$

as the consistency requires.

A natural $uosp(2|1)$ invariant measure on the supersphere $S^{2|2}$ can be defined with the help of the measure (10) on Hermitian supermatrices weighted by delta functions of all constraints which define the supersphere:

$$\begin{aligned} d\mu_{S^{2|2}} &:= d\mu_H \delta(\text{STr} \mathcal{M}) \delta\left(\frac{1}{2} \text{STr} \mathcal{M}^2 - 1\right) \delta(\mathcal{M}_{33}) \delta(\mathcal{M}_{23} - \bar{\mathcal{M}}_{13}) \delta(\bar{\mathcal{M}}_{23} + \mathcal{M}_{13}) = \\ &= dx_1 dx_2 dx_3 d\theta_+ d\theta_- \delta(x_1^2 + x_2^2 + x_3^2 + 2\theta_+ \theta_- - 1). \end{aligned} \quad (42)$$

Due to $uosp(2|1)$ invariance of the constraints, in order to check the invariance of the measure $d\mu_{S^{2|2}}$, it is sufficient to check the invariance of $d\mu_H$. The infinitesimal change of coordinates induced by the $uosp(2|1)$ element \mathcal{V} is obviously

$$\delta\mathcal{M} = i\varepsilon[\mathcal{V}, \mathcal{M}] \equiv i\varepsilon\text{Ad}_{\mathcal{V}}\mathcal{M} \quad (43)$$

where ε is a small parameter. The induced Berezinian is then

$$\text{sdet}(1 + i\varepsilon\text{Ad}_{\mathcal{V}}) = 1 + i\varepsilon\text{STr}(\text{Ad}_{\mathcal{V}}) = 1, \quad (44)$$

which means that the measure is indeed $uosp(2|1)$ invariant.

The immediate consequence of the invariance of the measure $d\mu_{S^{2|2}}$ is the formula

$$\int d\mu_{S^{2|2}}\{\mathcal{M}, f\} = 0, \quad \forall f \in C^{pol}(S^{2|2}), \quad (45)$$

since $\{\mathcal{M}, f\}$ is the (matrix valued) variation of the function f under (all possible) infinitesimal $uosp(2|1)$ transformations.

4 Sigma models

4.1 The bosonic case

Denote by y^I , $I = 1, \dots, n$ coordinates on the target Riemannian manifold T and, slightly abusing the notation, also the pull-backs ϕ^*y^I by some smooth map $\phi : S^2 \rightarrow T$. The bosonic sigma model action $\int d\mu_g ||d\phi||_{g,G}^2$ with the standard round metric g on S^2 and a metric $G_{IJ}(y^K)$ on T can be then rewritten in the following way (cf. Eq. (134) of [5]):

$$S_G = \int d\mu_{S^2} G_{IJ}(y^K) \mathcal{R}_m y^I \mathcal{R}_m y^J = \int d\mu_{S^2} G_{IJ}(y^K) \{x_m, y^I\} \{x_m, y^J\}. \quad (46)$$

Recall that here \mathcal{R}_m are the vector fields generating infinitesimal rotations of the sphere and the second equality in (46) follows from (24). Moreover, from (18) we have $M = x_m \sigma^m$ and we can therefore rewrite (46) in more invariant way as

$$S_G = \frac{1}{2} \text{Tr} \int d\mu_{S^2} G_{IJ}(y^K) \{M, y^I\} \{M, y^J\}. \quad (47)$$

It may appear natural to add to (47) the Kalb-Ramond term in its most symmetric form $\int \phi^* B$, however, such expression does not lend itself to the supersymmetric generalisation. We shall instead rewrite the full sigma model action $S_{GB} = \int d\mu_g ||d\phi||_{g,G}^2 + \int \phi^* B$ as

$$S_{GB} = \frac{1}{2} \text{Tr} \int d\mu_{S^2} (G_{IJ} + iM B_{IJ}) \{M, y^I\} \{M, y^J\} \equiv \text{Tr} \int d\mu_{S^2} L_{GB}, \quad (48)$$

where B_{IJ} are the components of the Kalb-Ramond form B in the coordinates y^I . The representation (48) of the full sigma model action was not obtained in [5], therefore we have to

justify it. For that we must first check the $so(3)$ invariance of the action (48) with respect to the infinitesimal rotations $\delta_V y^I = \{\text{Tr}(VM), y^I\}$. Using (20), we find successively

$$\delta_V G_{IJ}(y^K) = \{\text{Tr}(VM), G_{IJ}(y^K)\}, \quad (49)$$

$$\delta_V(B_{IJ}(y^K)M) = \{\text{Tr}(VM), B_{IJ}(y^K)\}M = \{\text{Tr}(VM), B_{IJ}(y^K)M\} - i[V, B_{IJ}(y^K)M], \quad (50)$$

$$\delta_V\{M, y^I\} = \{M, \delta_V y^I\} = \{\text{Tr}(VM), \{M, y^I\}\} - i[V, \{M, y^I\}], \quad (51)$$

$$\delta_V S_{GB} = \text{Tr} \int d\mu_{S^2} \{\text{Tr}(VM), L_{GB}\} - i \text{Tr} \int d\mu_{S^2} [V, L_{GB}] = 0. \quad (52)$$

Indeed, the last equality follows from (21) and from the fact that $\text{Tr}VL_{GB} = \text{Tr}L_{GB}V$. Now we are ready to verify that the Kalb-Ramond term $\int \phi^* B$ can be written as

$$\int \phi^* B = \frac{1}{2} \text{Tr} \int d\mu_{S^2} i M B_{IJ}(y^K) \{M, y^I\} \{M, y^J\}. \quad (53)$$

Indeed the integral of the differential form $\phi^*(dy^I \wedge dy^J)$ over S^2 can be certainly written as $\int d\mu_{S^2} < K, dy^I \wedge dy^J >$ where K is some bivector on S^2 (we write y^I instead of $\phi^* y^I$). Because S^2 is two-dimensional manifold, every two bivectors K and \tilde{K} are related as $\tilde{K} = fK$, where f is a function on the sphere. This means, in particular, that

$$\int \phi^* B = \frac{1}{2} \text{Tr} \int d\mu_{S^2} f i M B_{IJ}(y^K) \{M, y^I\} \{M, y^J\}. \quad (54)$$

However, the $so(3)$ invariance of both $\int \phi^* B$ and $\text{Tr} \int d\mu_{S^2} i M B_{IJ}(y^K) \{M, y^I\} \{M, y^J\}$ means that f must be a $so(3)$ invariant function hence a constant and it is easy to check that $f = 1$.

4.2 The supersymmetric case

In analogy with the bosonic case (48), it looks plausible that the action of the supersymmetric sigma model should be of the type

$$S_{tent} = \text{STr} \int d\mu_{S^{2|2}} (G_{IJ} + i \mathcal{M} B_{IJ}) \{\mathcal{M}, Y^I\} \{\mathcal{M}, Y^J\}. \quad (55)$$

Here Y^I are the sigma model superfields viewed as elements of $C^{pol}(S^{2|2})$, all other symbols were introduced in Section 3.2, and we wrote S_{tent} to indicate that this is just the tentative expression. Indeed, quite remarkably, this caution turns out to be fully justified since the tentative action (55) is pathological when worked out in components! (The problem is that upon the expansion in components the kinetic term for the fermions contains two bosonic derivatives.) It is therefore necessary to look for another expression and the main result of this article states that such a viable $uosp(2|1)$ -invariant action is in fact given by the following formula

$$S_{sGB} = -\text{STr} \int d\mu_{S^{2|2}} (G_{IJ} + 2i \mathcal{M} B_{IJ}) \{\mathcal{M}^2, Y^I\} \{\mathcal{M}^2, Y^J\}. \quad (56)$$

Before to argue that the action (56) is the correct one, let us show that for $B_{IJ} = 0$ it reduces to the supersymmetric action of Ref.[5]. For that, we first calculate \mathcal{M}^2 starting from (39) which gives

$$\mathcal{M}^2 = \begin{pmatrix} 1 - \theta_+ \theta_- & 0 & (x_1 - ix_2)\theta_+ - x_3\theta_- \\ 0 & 1 - \theta_+ \theta_- & -(x_1 + ix_2)\theta_- - x_3\theta_+ \\ (x_1 + ix_2)\theta_- + x_3\theta_+ & (x_1 - ix_2)\theta_+ - x_3\theta_- & -2\theta_+ \theta_- \end{pmatrix}. \quad (57)$$

Now we define three differential operators d_\pm and Γ via the Poisson brackets as

$$d_+ Y^I := \{(x_1 + ix_2)\theta_- + x_3\theta_+, Y^I\}, \quad d_- Y^I := \{(x_1 - ix_2)\theta_+ - x_3\theta_-, Y^I\}, \quad \Gamma Y^I := \{2\theta_+ \theta_-, Y^I\}. \quad (58)$$

Straightforward calculation using the Poisson brackets (37) and (38) reveals that the operators d_\pm and Γ coincide with those defined in Eqs. (50,51,52) of Ref.[5]. Now using the measure (42), substituting (57) in (56) and evaluating the supertrace we obtain for $B_{IJ} = 0$:

$$S_{sG} = 2 \int dx_1 dx_2 dx_3 d\theta_+ d\theta_- \delta(x_1^2 + x_2^2 + x_3^2 + 2\theta_+ \theta_- - 1) G_{IJ} (d_+ Y^I d_- Y^J - d_- Y^I d_+ Y^J + \frac{1}{4} \Gamma Y^I \Gamma Y^J). \quad (59)$$

This expression reduces (upon to integration per partes) to the Eq. (136) of Ref.[5].

There are three things that we have to verify in order to show that the action (56) is indeed the correct one in the presence of the B term. First of all it is $uosp(2|1)$ -invariance, then the correct bosonic limit and, thirdly, the absence of a quadratic expression in the bosonic derivatives in the fermionic part of the action.

1. The $uosp(2|1)$ -invariance of (56) is verified in the exactly same way (49,50,52) as in the bosonic case. Only Eq. (51) has a slightly different supersymmetric counterpart:

$$\delta_{\mathcal{V}} \{\mathcal{M}^2, Y^I\} = \{\mathcal{M}^2, \delta_{\mathcal{V}} Y^I\} = \{\text{STr}(\mathcal{V}\mathcal{M}), \{\mathcal{M}^2, Y^I\}\} - i[\mathcal{V}, \{\mathcal{M}^2, Y^I\}]. \quad (60)$$

2. In order to speak about the bosonic limit of (56), we must first embed $C^{pol}(S^2)$ in $C^{pol}(S^{2|2})$ and then to consider the action (56) evaluated at the configurations $\hat{y}^I \in C^{pol}(S^{2|2})$ which are the images of bosonic configurations $y^I \in C^{pol}(S^2)$ upon this embedding. The embedding itself was constructed in [5] and it is completely defined by the images $\hat{M}_{ij} \in C^{pol}(S^{2|2})$ (denoted by "hats") of the bosonic sphere generators $M_{ij} \in C^{pol}(S^2)$:

$$\hat{M}_{ij} = (\mathcal{M}_e(1 + \mathcal{M}_o^2))_{ij}, \quad i, j = 1, 2. \quad (61)$$

Here \mathcal{M}_e and \mathcal{M}_o are the even and the odd parts of the supermatrix \mathcal{M} and it is perhaps useful to rewrite (61) in components:

$$\hat{M}_{ij} = \mathcal{M}_{ij}(1 + \mathcal{M}_{13}\bar{\mathcal{M}}_{13}). \quad (62)$$

It is easy to check that (61) is consistent with the sphere and supersphere defining relations (11) and (31). Moreover, the embedding preserves the measure, i.e. it holds for every $f \in C^{pol}(S^2)$:

$$\int d\mu_{S^2} f = \int d\mu_{S^{2|2}} \hat{f}. \quad (63)$$

However, the Poisson structure is not completely preserved since it holds

$$\{\hat{f}, \hat{g}\}_{S^{2|2}} = \widehat{\{f, g\}}_{S^2} (1 + \mathcal{M}_{13} \bar{\mathcal{M}}_{13}), \quad f, g \in C^{pol}(S^2). \quad (64)$$

The crux of the argument is now based on the following identities

$$\{\mathcal{M}_e^2, \hat{y}\} = \{\mathcal{M}_o^2, \hat{y}\} = 0, \quad \{\mathcal{M}_o \mathcal{M}_e, \hat{y}\} = \frac{1}{2} \mathcal{M}_o \{\mathcal{M}_e, \hat{y}\}, \quad \{\mathcal{M}_e \mathcal{M}_o, \hat{y}\} = \frac{1}{2} \{\mathcal{M}_e, \hat{y}\} \mathcal{M}_o, \quad (65)$$

where $\hat{y} \in C^{pol}(S^{2|2})$ is the embedding of some $y \in C^{pol}(S^2)$. The identities (65) can be verified straightforwardly by setting successively $y = x_m, \theta_\pm$, inserting the explicit expressions (39) and (57) for \mathcal{M} and \mathcal{M}^2 and evaluating the obtained Poisson brackets using (37) and (38).

We can now start to evaluate the action S_{sGB} on the bosonic sigma model configuration $y^K \in C^{pol}(S^2)$ embedded in $C^{pol}(S^{2|2})$ as \hat{y}^K :

$$\begin{aligned} S_{sGB}(\hat{y}^K) &= -\text{STr} \int d\mu_{S^{2|2}} (G_{IJ} + 2i\mathcal{M}B_{IJ}) \{\mathcal{M}^2, \hat{y}^I\} \{\mathcal{M}^2, \hat{y}^J\} = \\ &= -\text{STr} \int d\mu_{S^{2|2}} (G_{IJ} + 2i\mathcal{M}B_{IJ}) \{\mathcal{M}_e \mathcal{M}_o + \mathcal{M}_o \mathcal{M}_e, \hat{y}^I\} \{\mathcal{M}_e \mathcal{M}_o + \mathcal{M}_o \mathcal{M}_e, \hat{y}^J\} = \\ &= -\frac{1}{4} \text{STr} \int d\mu_{S^{2|2}} (G_{IJ} + 2i\mathcal{M}_e B_{IJ}) (\mathcal{M}_o \{\mathcal{M}_e, \hat{y}^I\} \{\mathcal{M}_e, \hat{y}^J\} \mathcal{M}_o + \{\mathcal{M}_e, \hat{y}^I\} \mathcal{M}_o^2 \{\mathcal{M}_e, \hat{y}^J\}) = \\ &= -\frac{1}{2} \text{STr} \int d\mu_{S^{2|2}} \mathcal{M}_o^2 (G_{IJ} + i\hat{M}B_{IJ}) \{\hat{M}, \hat{y}^I\} \{\hat{M}, \hat{y}^J\} = \frac{1}{2} \text{Tr} \int d\mu_{S^2} (G_{IJ} + iMB_{IJ}) \{M, y^I\} \{M, y^J\}. \end{aligned} \quad (66)$$

In deriving (66) we have used (63), (64), the fact that $\mathcal{M}_o \mathcal{M} \mathcal{M}_o = 0$, that \mathcal{M}_o^2 commutes with all matrices in the r.h.s. of (66) and also the facts like e.g. $\{\mathcal{M}_e \mathcal{M}_o, \hat{y}^I\} \{\mathcal{M}_e \mathcal{M}_o, \hat{y}^J\}$ vanishes being the product of two odd upper-triangular matrices. Moreover, the last equality in (66) is obtained by integrating over the odd generators $\mathcal{M}_{13}, \bar{\mathcal{M}}_{13}$ which are present only in the matrix \mathcal{M}_o^2 since at every other place, including the measure delta function $\delta(\mathcal{M}_{11}^2 + \mathcal{M}_{12} \bar{\mathcal{M}}_{12} + 2\mathcal{M}_{13} \bar{\mathcal{M}}_{13} - 1)$, they are killed by the nilpotency.

3. It remains to verify the absence of a quadratic bosonic derivatives in the fermionic part of the action (56). For that we need not enter too far into the jungle of component calculations. We just consider the fermionic part Y_o^I of the superfield Y^I and we can write it as

$$Y_o^I(\mathcal{M}) = \Psi_-^I(\hat{M}) \mathcal{M}_{13} - \Psi_+^I(\hat{M}) \bar{\mathcal{M}}_{13}, \quad (67)$$

where the matrix \hat{M} was defined in (61) and the reality of the superfield $Y_o^I(\mathcal{M})$ is ensured by requiring $\bar{\Psi}_+ = \Psi_-$. Now we study the expression $\{\mathcal{M}^2, Y_o^I\} = \{\mathcal{M}^2, \Psi_-^I(\hat{M}) \mathcal{M}_{13} - \Psi_+^I(\hat{M}) \bar{\mathcal{M}}_{13}\}$ appearing in the action (56). We find from Eq. (14) that the components of the even part $(\mathcal{M}^2)_e$ of the supermatrix \mathcal{M}^2 Poisson-commute with the components of \mathcal{M}_e . This means that only the Poisson bracket $\{(\mathcal{M}^2)_o, Y_o^I\}$ can contain the bosonic derivatives $\{\mathcal{M}_e, \Psi_\pm^I\}$. By using Eq. (65), we obtain

$$\{(\mathcal{M}^2)_o, \Psi_\pm^I(\hat{M})\} = \{\mathcal{M}_o \mathcal{M}_e + \mathcal{M}_e \mathcal{M}_o, \Psi_\pm^I(\hat{M})\} = \frac{1}{2} \mathcal{M}_o \{\mathcal{M}_e, \Psi_\pm^I\} - \frac{1}{2} \{\mathcal{M}_e, \Psi_\pm^I\} \mathcal{M}_o \quad (68)$$

hence

$$\{(\mathcal{M}^2)_o, Y_o^I\} = -\Psi_-^I(\hat{M})\{(\mathcal{M}^2)_o, \mathcal{M}_{13}\} + \Psi_+^I(\hat{M})\{(\mathcal{M}^2)_o, \bar{\mathcal{M}}_{13}\} + \mathcal{M}_{13}\bar{\mathcal{M}}_{13}V \quad (69)$$

This means that the bosonic derivatives of fermions $\{\mathcal{M}_e, \Psi_\pm^I\}$ appear only in the expression V that multiplies $\mathcal{M}_{13}\bar{\mathcal{M}}_{13}$. The fact that $\mathcal{M}_{13}\bar{\mathcal{M}}_{13}$ squares to zero thus excludes a presence of the quadratic bosonic derivatives in the fermionic part of the action.

4.3 Supersymmetric sigma model on $S^{2|2}$ in components

There is an important question to be asked about the superfield action (56): Does it reduce to the flat action (3) in the weakly curved region when the radius of the supersphere approaches infinity? There is an indirect argument showing that this is true. Indeed, in this limit the $uosp(2|1)$ supersymmetry algebra gets contracted to the standard flat Euclidean supersymmetry algebra thus the limiting action must be supersymmetric in the Euclidean sense. Moreover, as it is well-known, the Kalb-Ramond term described in (3) is the only one respecting the criteria of the flat supersymmetry and the correct bosonic limit. So, said in other words, the large supersphere radius limit of (56) must give (3).

We did not find a direct check of the correct large radius limit other than a comparison of the component actions respectively derived from the superfield actions (3) and (56). The straightforward albeit tedious calculation is performed starting from the component version of (56) which we work out via the ansatz

$$Y^I(\mathcal{M}) = \hat{y}^I + \Psi_-^I \mathcal{M}_{13} - \Psi_+^I \bar{\mathcal{M}}_{13} + F^I \mathcal{M}_{13} \bar{\mathcal{M}}_{13}. \quad (70)$$

By eliminating the auxiliary fields F^I , we obtain

$$\begin{aligned} S_{sGB} = & \frac{1}{2} \int d\mu_{S^2} \text{Tr} \left[(G_{IJ} + iMB_{IJ}) \{M, y^I\} \{M, y^J\} + 2G_{IJ} (i\{M, \Psi^J\} + \Psi^J) \bar{\Psi}^I \right] + \\ & + \int d\mu_{S^2} \left[\bar{\Psi}^I i\{M, y^K\} (\Gamma_{IKL} + iMH_{IKL}) \Psi^L - \frac{1}{8} \mathcal{R}_{IJKL} (\bar{\Psi}^I \Psi^K - \bar{\Psi}^I M \Psi^K) (\bar{\Psi}^J \Psi^L - \bar{\Psi}^J M \Psi^L) \right], \end{aligned} \quad (71)$$

where

$$\Psi^I := \begin{pmatrix} \Psi_+^I \\ \Psi_-^I \end{pmatrix}, \quad \bar{\Psi}^I := (\bar{\Psi}_+^I \quad \bar{\Psi}_-^I) = (\Psi_-^I \quad -\Psi_+^I) \quad (72)$$

and the notation $\{M, \Psi^J\}$ means at the same time the Poisson bracket and the matrix action on a column vector:

$$\{M, \Psi\}_\alpha := \sum_\beta \{M_{\alpha\beta}, \Psi_\beta\}. \quad (73)$$

Moreover, the quantities Γ_{IKL} , H_{IKL} and \mathcal{R}_{IJKL} are defined as

$$\begin{aligned} \Gamma_{IKL} &= \frac{1}{2} (\partial_L G_{KI} + \partial_K G_{IL} - \partial_I G_{KL}), \quad H_{IKL} = \frac{1}{2} (\partial_I B_{KL} + \partial_K B_{LI} + \partial_L B_{IK}), \\ \mathcal{R}_{IJKL} &:= G_{IM} \mathcal{R}_{JKL}^M, \quad \mathcal{R}_{JKL}^M := \partial_K E_{LJ}^M - \partial_L E_{KJ}^M + E_{KN}^M E_{LJ}^N - E_{LN}^M E_{KJ}^N, \end{aligned} \quad (74)$$

$$E_{LJ}^K := G^{KN}(\Gamma_{NLJ} + iH_{NLJ}). \quad (75)$$

We recognize in the quantity $G^{KN}\Gamma_{NLJ}$ the standard Christoffel symbol corresponding to the metric G_{IJ} , the totally antisymmetric tensor H_{IJK} is nothing but the exterior derivative of the two-form B_{IJ} and \mathcal{R}_{IJKL} are the components of the modified Riemann curvature tensor corresponding to the connection E_{LJ}^K containing the torsion part H_{LJ}^K .

We notice that the component action (71) has again the elegant property that, apart from the dynamical fields y^I and Ψ^I , it contains just the Poisson brackets and the moment map M . However, we have to admit that in this particular case we were not able to preserve the elegance in all intermediate calculations. Indeed, while everything else in this paper was computed very directly and effortlessly thanks to our invariant Poisson language, the formula (71) was worked out by a tedious component calculation.

5 Conclusions and outlook

Apart from our main result which is the construction of the action of the $UOSp(2|1)$ supersymmetric sigma model with the Kalb-Ramond term, the present article also offers a conceptual simplification and a technical streamlining of the results of the reference [5]. In particular, all five generators R_3, R_\pm, V_\pm of the Lie superalgebra $uosp(2|1)$ and all three supersymmetric covariant derivatives Γ, D_\pm appearing explicitly in majority of formulas of [5] are conveniently arranged as matrix elements of a single supermatrix \mathcal{M} and its square \mathcal{M}^2 . Moreover, all calculations of [5] can be rephrased in terms of the matrices \mathcal{M} and \mathcal{M}^2 as a whole without a necessity to manipulate the matrix elements themselves. As for the outlook, we expect that the construction of supersymmetric gauge theories on the supersphere presented in [12] could be equally streamlined and rendered conceptually more transparent by using the moment map supermatrix \mathcal{M} .

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